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THE PROBLEM OF A PENDULUM WITH AN OSCILLATING POINT OF SUSPENSION†

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A global qualitative investigation of the equation of a pendulum with a vertically oscillating point of suspension in the non-conservative case close to the non-linear integrable case, is presented. The transition from non-linear resonance to parametric resonance when the frequency of oscillation of the point of suspension is changed is analysed. The behaviour of the solutions both in the oscillatory and rotational regions and also in the neighbourhood of the unperturbed separatrice is considered. The condition for a quasi-attractor to exist is established. The results of a numerical analysis, which agree with the theoretical results and illustrate them, are presented.

A pendulum with an oscillating point suspension is a classical example of a problem in which parametric resonance is observed. A considerable number of papers have been published on this problem (for example, [1, 2]). We also note problems on the flexural oscillations of a straight rod loaded with a periodic longitudinal force [3], the motion of a charged particle (an electron) in the field of two travelling waves [4], etc. The occurrence of parametric resonance in such systems is related to the loss in stability of a moving point of the corresponding Poincaré mapping and is therefore usually described by a system that is linearized in the neighbourhood of this point.

Problems of the existence and stability of resonant periodic motions have recently been solved (see, for example, [1, 2, 5]). There is also a theory which enables the global behaviour of the solutions of such systems to be investigated in the quasi-integrable case in regions not containing states of equilibrium and the separatrices of the unperturbed system [6–8]. An analysis of the resonance zones occupies a central place in this theory. It is of interest to investigate the behaviour of the parametric system when the resonance ring zone contracts to a point, i.e. bifurcations are established which originate when a transition occurs from the usual non-linear resonance to parametric resonance. This problem was considered in [9] for non-parametric systems. Local rearrangements of the phase pattern of truncated systems ("principal deformations of q-equivariant vector fields") have also been investigated [10] in the neighbourhood of this point. The present paper is also devoted to solving this problem using the example of a non-conservative pendulum with a vertically oscillating point of suspension.

In the quasi-integrable extremely non-linear case we also solve the problem of the motion of a pendulum in global regions (both oscillatory and rotational) and in the neighbourhood of the unperturbed separatrice, i.e. a global investigation is carried out of a pendulum with an oscillating point of suspension. A similar investigation was carried out for Duffing's equation [11] (see also [6]). Hence, only the main features are examined in detail in the investigation.

1. FORMULATION OF THE PROBLEM. THE QUASI-INTEGRABLE CASE

The equation of motion of a pendulum with a vertically oscillating point of suspension has the form, with certain simplifying assumptions,

$$(J + ma^2)\theta^{\cdot\cdot} + mar\nu^2[g/(r\nu^2) + \cos \nu t] \sin \theta + \delta\theta^{\cdot} = 0 \tag{1.1}$$

where J is the moment of inertia of the pendulum about the axis passing through the centre of mass perpendicular to the plane of oscillations, δ is the coefficient of viscous friction, m is the mass of the pendulum, a is its length, r is the amplitude of the oscillations of the point of suspension, g is the acceleration due to gravity and θ is the angle of deflection of the pendulum from the position of equilibrium.

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Making the replacement of time $\sqrt{(mag/(J + ma^2))}\, t \Rightarrow t$ and of the coordinate $\theta \Rightarrow x$ in (1.1) we obtain the equation

$$x^{\cdot\cdot} + \sin x + p_1 \cos\beta t \sin x + p_2 x^{\cdot} = 0 \tag{1.2}$$

$$p_1 = rv^2/g,\ p_2 = \delta/\sqrt{mag(J+ma^2)},\ \beta = v\sqrt{(J+ma^2)/(mag)}$$

We will complicate the model further and consider the equation

$$x^{\cdot\cdot} + \sin x + p_1 \cos\beta t \sin x + (p_2 + p_3 \cos x) x^{\cdot} = 0 \tag{1.3}$$

the phase space of which is $\mathbb{R}^1 \otimes S^1 \otimes S^1$. The term $p_3 \cos(x) x$ appears, for example, in the case of a pendulum in which the resistance force is produced by a vertically oriented plate perpendicular to the plane of oscillations. Note also that for a mathematical pendulum $\beta = v\sqrt{(a/g)}$.

Equation (1.3) is not amenable to an analytic global investigation for arbitrary parameters. An equation of the form (1.2) has been investigated analytically in the quasi-linear case (see, for example, [1, 2]), and local problems of the existence and stability of periodic motions have been solved (see, for example, [2, 5]), as well as the problem of the existence of an irregular structure or doubly asymptotic solutions (see, for example, [4, 12–15]). However there are no publications giving a global analysis of Eq. (1.3) in the extremely non-linear quasi-integrable case (when the parameters p_i are small). In this paper we attempt to fill this gap.

Consider Eq. (1.3) in the case close to integrable, i.e. for small values of the parameters p_i ($i = 1, 2, 3$). We will put $p_i = -\varepsilon C_i$, where ε is a small parameter. Then the initial equation (1.3) takes the form

$$x^{\cdot\cdot} + \sin x = \varepsilon[C_1 \cos\beta t \sin x + (C_2 + C_3 \cos x) x^{\cdot}] \tag{1.4}$$

It is clear that in (1.4) we can dispense with one parameter and consider a two-parametric family of vector fields. However, for convenience we will not do this.

An equation of the form (1.4) we can dispense with one parameter and consider a two-parametric family of vector fields. However, for convenience we will not do this.

An equation of the form (1.4) in the conservative case when $C_2 = C_3 = 0$, has been considered in many publications. Thus, the case when $\beta \cong 2$ for small angles of deflection x was considered in [1]. The criterion for resonances to overlap was used in [4] to estimate the width of the "ergodic layer", and the existence of doubly asymptotic (homoclinic) solutions was investigated in [13, 14]. A complete analysis of this problem was given in [14] without assuming the parameter ε to be small. The existence of homoclinic solutions in Eq. (1.4) is an obstacle to its integrability [14, 15]. The problem of the existence of limit cycles and quasi-attractors in an equation similar in form to (1.4) was solved in [16].

We know that the unperturbed equation $x^{\cdot\cdot} + \sin x = 0$ allows of a first integral (the energy integral)

$$H(x,y) \equiv y^2/2 - \cos x = h = \text{const},\ y = x^{\cdot}$$

Oscillatory motions of the pendulum correspond to the values $h \in (-1, 1)$, while rotational motions correspond to values of $h > 1$. A feature of the equation of a mathematical pendulum is the fact that the period τ depends on h in the oscillatory region

$$\tau(h) = 4K(k),\ k^2 = \rho = (1+h)/2,\ -1 < h < 1 \tag{1.5}$$

Here $K = K(k)$ is the complete elliptic integral of the first kind and k is its modulus.

It follows from (1.5) that the period τ changes considerably only for values of h close to unity, i.e. in the neighbourhood of the separatrice. Hence, small intervals with respect to the period τ, defining the width of the resonance zone, lead to fairly large intervals with respect to the coordinate x. It should also be recalled that the value of the natural frequency $\omega(h) = 2\pi/\tau$ when $h = -1$ is equal to unity, whereas for the initial equation it is equal to $\sqrt{(mag/(J + ma^2))}$.

2. THE STRUCTURE OF THE RESONANCE ZONES

When investigating the perturbed equation we will dwell primarily on the problem of the structure of the resonance zones situated in the regions $G^1 = \{(x, \dot{x}): -1 < h_- \leqslant H(x, y) \leqslant h_+ < 1\}$ and

$G^2 = \{(x, \dot{x}): H(x, y) \geqslant h_* > 1\}$. The condition for resonance $\tau(h_{pq}) = (p/q)(2\pi/\beta)$, where p and q are relatively prime integers, determines the resonance energy levels: $H(x, y) = h_{pq}$.

The structure of the individual resonance zones

$$U_{\sqrt{\varepsilon}} = \{(x, \dot{x}):\ h_{pq} - \sqrt{\varepsilon}C < H(x, y) < h_{pq} + \sqrt{\varepsilon}C\},\quad C = \text{const}$$

is described (to terms of the order of $O(\varepsilon^{3/2})$) by a pendulum-type equation [6–8]

$$\frac{d^2 \upsilon}{d\tau^2} - bA(\upsilon; h_{pq}) = \mu\sigma(\upsilon; h_{pq})\frac{d\upsilon}{d\tau} \tag{2.1}$$

Here

$$A(\upsilon; h_{pq}) = \frac{1}{2\pi p}\int_0^{2\pi p} F\left(h_{pq}, \upsilon + \frac{q\varphi}{p}, \varphi\right) d\varphi \tag{2.2}$$

$$\sigma(\upsilon; h_{pq}) = \frac{1}{2\pi p}\int_0^{2\pi p} \frac{\partial f(x, \dot{x}, \varphi)}{\partial \dot{x}} d\varphi \tag{2.3}$$

and $x = x(\upsilon + q\varphi/p; h_{pq})$, $\dot{x} = \dot{x}(\upsilon + q\varphi/p; h_{pq})$ is the solution of the unperturbed equation at the level $H(x, \dot{x}) = h_{pq}$.

We used the action (I)–angle (ϑ), $\vartheta = \upsilon + q\varphi/p$ variables when deriving (2.1).

The unperturbed solution for the oscillatory region ($-1 < h < 1$) and the rotational region ($h > 1$) has the form

$$x(\vartheta) = 2\arcsin\left(k\,\mathrm{sn}\frac{2K\vartheta}{\pi}\right),\quad \dot{x} = y = 2k\,\mathrm{cn}\frac{2K\vartheta}{\pi},\quad \vartheta = \omega t,\quad \omega = \frac{\pi}{2K},\quad -1 < h < 1 \tag{2.4}$$

$$x(\vartheta) = 2\,\mathrm{am}\frac{K\vartheta}{\pi},\quad y = \pm\frac{2}{k}\mathrm{dn}\frac{2K\vartheta}{\pi},\quad \omega = \frac{\pi}{kK},\quad k^2 = \frac{2}{1+h},\quad h > 1$$

Since the functions A and σ are different in the oscillatory and the rotational regions, we will introduce the notation $A^{(s)}(\upsilon, h_{pq})$, $\sigma^{(s)}(\upsilon, h_{pq})$, where $s = 1$ corresponds to the oscillatory region and $s = 2$ corresponds to the rotational region.

The functions $A^{(s)}$, $\sigma^{(s)}$ are periodic in υ with the least period $2\pi/p$ [6]. Since in the case considered the divergence of the vector field of Eq. (1.4) does not contain terms which depend explicitly on the time t, the quantity σ is independent of υ [17], i.e. $\sigma = \text{const}$.

We will calculate $A^{(1)}$ and $\sigma^{(1)}$. It follows from (2.2) and the results obtained previously [16] that

$$A^{(1)} = 2C_1\frac{\rho}{\pi\omega q}\int_0^{2\pi q}\cos\frac{p(\vartheta - \upsilon)}{q}\,\mathrm{sn}\frac{2K\vartheta}{\pi}\,\mathrm{dn}\frac{2K\vartheta}{\pi}\,\mathrm{cn}\frac{2K\vartheta}{\pi}\,d\vartheta + \frac{8}{q\pi}[C_2F_0^{(1)}(\rho) + C_3F_1^{(1)}(\rho)] \tag{2.5}$$

$$F_0^{(1)} = (\rho - 1)K(k) + E(k),\quad F_1^{(1)} = [(1-\rho)K(k) + (2\rho - 1)E(k)]/3,\quad \rho = k^2$$

where $E = E(k)$ is the complete elliptic integral of the second kind. Evaluation of the integral in (2.5) gives

$$A^{(1)} = C_1\Gamma_p^{(1)}\sin(p\upsilon) + B^{(1)} \tag{2.6}$$

$$\Gamma_p^{(1)} = \begin{cases} 0 \text{ for } p \neq 2(2n-1) \text{ and/or } q > 1 \\ 8\beta^2 a_{p/2}^2/p \quad \text{for } p = 2(2n-1) \text{ and } q = 1,\ n = 1, 2, \ldots \end{cases}$$

$$a_{p/2} = \frac{\alpha^{p/4}}{1 - \alpha^{p/2}},\quad \alpha = \exp\left(-\frac{\pi K(\sqrt{1-\rho})}{K(\sqrt{\rho})}\right),\quad B^{(1)} = \frac{8}{\pi}[C_2F_0^{(1)} + C_3F_1^{(1)}],$$

where $B^{(1)}$ is the generating Poincaré–Pontryagin function for the autonomous equation ($C_1 = 0$; for more detail on the generating function see [6, 7]).

From (2.3) we obtain

$$\sigma^{(1)}(\upsilon, h_{pq}) = C_2 + C_3(2E - K)/K$$

Similar calculations for the rotational region give

$$A^{(2)} = C_1\Gamma_p^{(2)} \sin p\upsilon + B^{(2)} \tag{2.7}$$

where p is an odd number and

$$\Gamma_p^{(2)} = \begin{cases} 0 & \text{for} \quad p \neq 2n-1 \quad \text{and/or} \quad q > 1 \\ 4\beta^2 a_p^2 / p & \text{for} \quad p = 2n-1 \quad \text{and} \quad q = 1, \quad n = 1, 2, \ldots \end{cases}$$

$$a_p = \frac{\alpha^{p/2}}{1 - a^p}, \quad B^{(2)} = \frac{8}{\rho^{1/2}\pi}[C_2 F_0^{(2)} + C_3 F_1^{(2)}]$$

$$F_0^{(2)} = E(\rho), \quad F_1^{(2)} = [2(\rho - 1)K + (2 - \rho)E]/(3\rho)$$

$$\sigma^{(2)}(\upsilon, h_{pq}) = C_2 + C_3((p-2)K + 2E)/(\rho K)$$

It follows from (2.6)–(2.7) that the width of the resonance zone, defined by the quantity $C_1\Gamma_p^{(s)}$, falls rapidly as p increases. Hence, when $C_2^2 + C_3^2 \neq 0$ it is difficult to detect resonance modes with $p > 2$ in the oscillatory region and with $p > 1$ in the rotational region. The condition for a resonance mode with $p = 2$ to exist by (2.6) has the form $|C_1\Gamma^{(s)}| > |B^{(1)}|$. For example, for $\beta = 1.6$ it reduces to the condition $|C_1| > a_1C_2 + a_2C_3$, $a_1 \simeq 9.71$, $a_2 \simeq 6.41$.

The change from the truncated system (2.1) to the initial system (1.4) gives a well-known result: if the truncated system (2.1) is rough in the Andronov–Pontryagin sense, then small corrections which manifest themselves when changing to the initial system do not change the behaviour of the solutions to any great extent (see, for example, [7, 11] for more detail).

By virtue of the condition $\sigma^{(s)}(\rho_*) \neq 0$ when $B^{(s)}(\rho_*) = 0$, Theorem 1 from [8], which defines the global qualitative behaviour of the solutions in the regions $G^{(s)}$ ($s = 1, 2$), holds.

The behaviour of the invariant curves of the Poincaré mapping for Eq. (1.3), obtained on a computer, is shown in Figs 1 and 2 for $\beta = 1.6$. In Fig. 1(a) we show the case of synchronization of the oscillations at the subharmonic with $p = 2$ ($B(h_{21}) = 0$; $p_1 = 0.1, p_2 = 0.07, p_3 = -0.1$), and in Fig. 1(b) we show the partially traversed resonance with $p = 2$ ($p_1 = 0.1, p_2 = 1/30, p_3 = -0.1$). In Fig. 2 we show the behaviour of the invariant curves of the Poincaré mapping in the rotational region in the upper half-cylinder ($x \bmod 2\pi, y; y > 0$) for Eq. (1.3) for $p_1 = 0.1, p_2 = 0.015$ and $p_3 = -0.1$ (the pattern of the behaviour of the invariant curves on the lower half-cylinder ($y < 0$) is symmetrical to the pattern of their behaviour on the upper half-cylinder). In this case synchronization of the oscillations occurs on axial resonance ($p = 1, q = 1$). The resonance zone for the fundamental resonance is situated in the region of the separatrices of the fixed point $(\pi, 0)$. As the frequency β increases the resonance level $H(x, y) = h_{11}$ is raised upwards along the cylinder in accordance with the formula for the natural frequency and the resonance condition. General agreement with the theoretical results can be seen. However, the averaged system (2.1) does not determine the irregular way in which the resonance mode becomes established, shown in Fig. 2. The arrows in Figs 1 and 2 indicate the direction of motion as t increases.

Note that by (2.1), (2.6) and (2.7), a change in the sign of the parameter p will change the saddle points in the resonance zones to "stable" points, while the "stable" points will change into saddle points. Thus, in the case of resonance with $p = 2$ and $q = 1$ this leads to rotation of the pattern of the behaviour of the invariant curves of the Poincaré mapping in the resonance zone by 90° in a clockwise direction.

3. THE NEIGHBOURHOOD OF THE ORIGIN OF COORDINATES

We put $U_n = \{(x, y): 0 \leqslant H(x, y) \leqslant C\varepsilon^{2/n}\}$ and make the following change in Eq. (1.4)

$$x = \varepsilon^{1/n}\xi, \quad y = x^{\cdot} = \varepsilon^{1/n}\eta$$

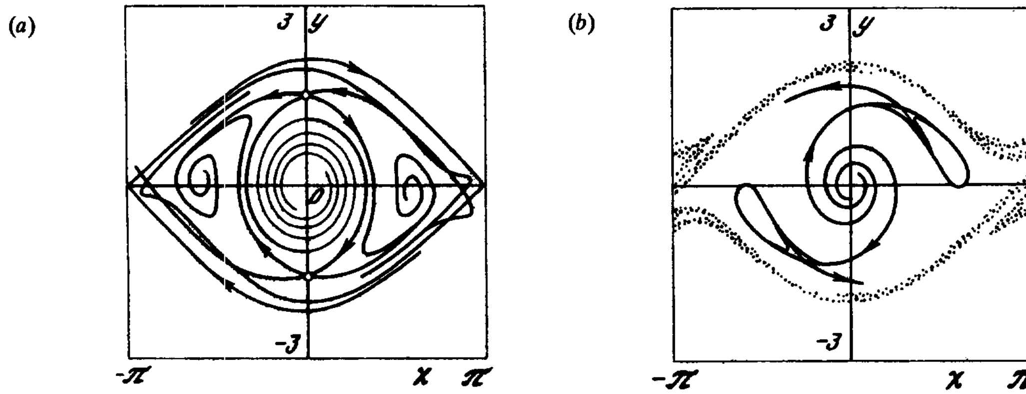

Fig. 1.

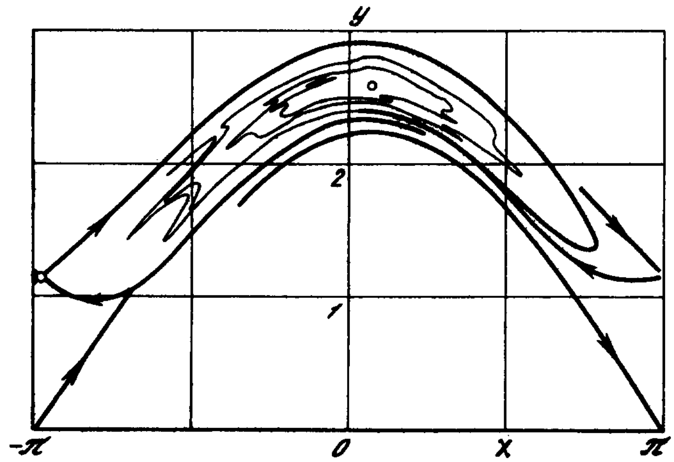

Fig. 2.

As a result we obtain the system

$$\xi^{\cdot} = \eta, \ \ \eta^{\cdot} = -\xi + \varepsilon[C_1\xi\cos\beta t + (C_2 + C_3)\eta] + \varepsilon^{2/n}\xi^3/6 - \varepsilon^{1+2/n}(C_1\xi^3\cos\beta t/6 + \xi^2\eta) + \ldots \tag{3.1}$$

System (3.1) is defined in $D \otimes S^1$, where D is a certain region from $\mathbb{R}^2$.
In the neighbourhood if $U_1(n = 1)$ system (3.1) takes the form

$$\xi^{\cdot} = \eta, \ \ \eta^{\cdot} = -\xi + \varepsilon[C_1\xi\cos\beta t + (C_2 + C_3)\eta] + O(\varepsilon^2) \tag{3.2}$$

If we neglect terms of the order of $O(\varepsilon^2)$ in (3.2) we obtain a Mathieu equation with an additional term which takes viscous friction into account. Clearly, using the linear equation, we cannot obtain non-linear effects related to the transition from non-linear resonance to parametric resonance. We will therefore consider a wider neighbourhood U_2 $(n = 2)$ of the origin of coordinates. In this case, neglecting terms $O(\varepsilon^2)$ in (3.1) we obtain a system for which we will consider the resonance cases when $\omega = 1 = q\beta/p$, where p and q are relatively prime integers. To investigate the bifurcations connected with the transition from parametric resonance to ordinary resonance we will introduce the detuning $1 - q\beta/p = \gamma_1\varepsilon$. As a result, the system considered can be rewritten in the form

$$\begin{aligned} \xi^{\cdot} &= (q\beta/p)\eta + \gamma_1\varepsilon \\ \eta^{\cdot} &= -(q\beta/p)\xi + \varepsilon[C_1\xi\cos\beta t + (C_2 + C_3)\eta - \gamma_1\xi + \xi^3/6] \end{aligned} \tag{3.3}$$

We used the action (I)–angle(ϑ) variables [6, 7] when deriving Eq. (2.1). We will also use these variables here. Since the unperturbed system is linear, this replacement takes the simple form $\xi = \sqrt{(2I)}\sin\vartheta$,

$\eta = \sqrt{(2I)}\cos\vartheta$. After this replacement, system (3.3) can be written in the form

$$I^{\cdot} = \varepsilon F(I,\vartheta,\varphi),\ \ \vartheta^{\cdot} = q\beta/p - \varepsilon R(I,\vartheta,\varphi),\ \ \varphi^{\cdot} = \beta \tag{3.4}$$
$$F = 2IG\cos\vartheta - \gamma_1\sqrt{2I}\sin\vartheta,\quad R = G\sin\vartheta + \gamma_1\cos\vartheta/\sqrt{2I}$$
$$G = C_1\sin\vartheta\cos\varphi + (C_2+C_3)\cos\vartheta - \gamma_1\sin\vartheta + (1/3)\sin^3\vartheta$$

We will introduce the "resonance phase" $\psi = \upsilon - q\varphi/p$ in (3.4) and average the system obtained over one "fast" variable φ. As a result we obtain the following two-dimensional autonomous system

$$u^{\cdot} = \varepsilon[(C_1/2)u\sin(2\upsilon) + (C_2+C_3)u] \tag{3.5}$$
$$\upsilon^{\cdot} = \varepsilon[(C_1/4)\sin(2\upsilon) - u/8 - \gamma_1/2]$$

for $p = 2$ and $q = 1$, and the system

$$u^{\cdot} = \varepsilon(C_2+C_3)u,\quad \upsilon^{\cdot} = \varepsilon(-u/8 - \gamma_1/2) \tag{3.6}$$

for $p \neq 2$ and/or $q > 1$. As we know [6], $u = I + O(\varepsilon)$, $\upsilon = \psi + O(\varepsilon^2)$. It follows from (3.5) and (3.6) that in the neighbourhood of U_2 in this approximation there is only one resonance with $p = 2$ and $q = 1$.

We now return to system (3.5). The cylinder $\{\upsilon \bmod (\pi), u\}$ serves as the phase space of system (3.5). This system is a Hamiltonian system when $C_2 = C_3 = 0$, $\gamma_1 = 0$ with Hamiltonian $H(u, \upsilon) = -\varepsilon C_1 u \cos(2\upsilon)/4 + \varepsilon u^2/16$.

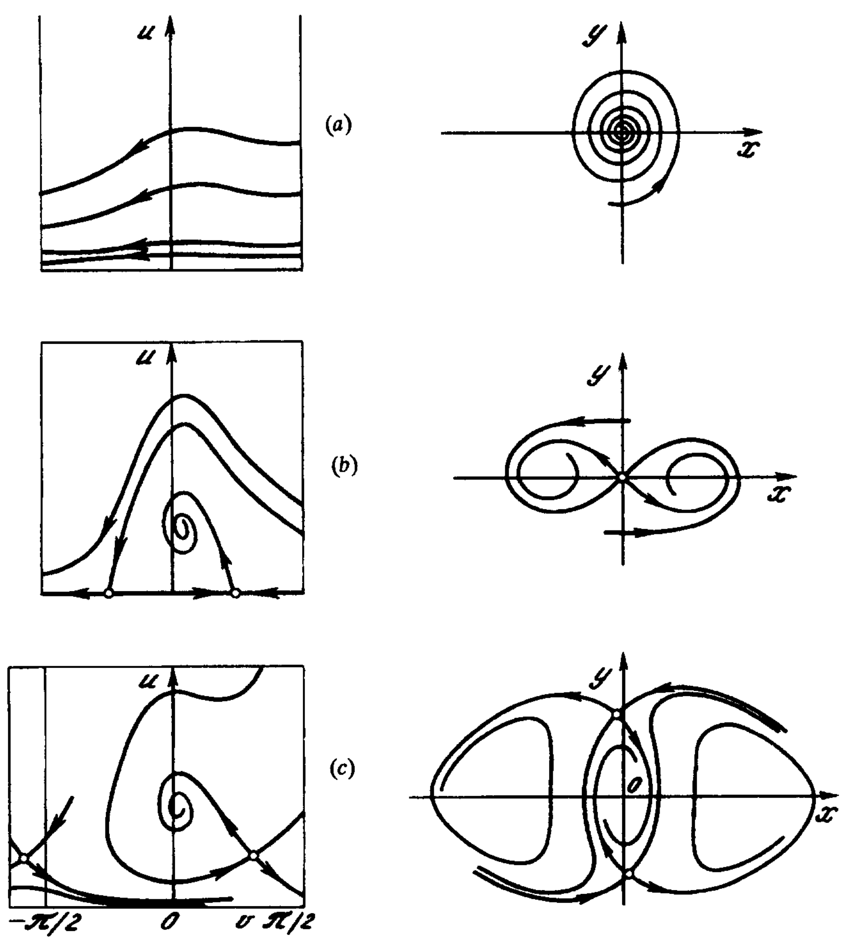

Fig. 3.

It is not particularly difficult to investigate system (3.5) when $C_2^2 + C_3^2 \neq 0$ and for different values of the detuning γ_1 since, by the Bendixon criterion, there are no limit cycles. The most characteristic rough phase patterns (in the Andronov–Pontryagin sense) are shown in Fig. 3, where in addition to the phase patterns in the (u, υ) plane we also show the corresponding phase patterns in Cartesian coordinates $(x, y = \dot{x})$. Figure 3(a) corresponds to the case when $\gamma_1 > \gamma_* > 0$, $\gamma_* = \sqrt{(C_1^2 - 4(C_2 + C_3)^2)}/2$ in system (3.5), Fig. 3(b) corresponds to the case when $|\gamma_1| \leqslant \gamma_*$, and Fig. 3(c) corresponds to the case when $|\gamma_1| > \gamma_*$, $\gamma_1 < 0$. In addition, in all cases $C_2 + C_3 < 0$.

When the parameter ε is not small, the Poincaré mapping in the neighbourhood of the origin of coordinate leads to phase patterns in the (x, y) plane corresponding to Fig. 143 in [10] for the "main deformation of the 2-equivalent field". Note that the main deformations from [10] do not give the phase pattern shown in Fig. 3(c) and its modifications when there is a limit cycle. The latter corresponds to the passage of a cycle through the resonance zone [7, 8] and is naturally not described by the classical local theory.

4. THE NEIGHBOURHOOD OF THE SEPARATRICE

As we know, a homoclinic Poincaré structure and the related irregular behaviour of the solutions can exist in the neighbourhood of the unperturbed separatrice. The condition for such a structure to exist can be obtained using the Mel'nikov function [18] $\Delta_\varepsilon(t') = \varepsilon\Delta_1(t') + O(\varepsilon^2)$, where

$$\Delta_1(t') = E\sin(\beta t') + E_*;\quad E = -\frac{2C_1\pi\beta^2}{\mathrm{sh}(\pi\beta/2)},\quad E_* = 4\pi\left(C_2 + \frac{C_3}{3}\right)$$

Here we have used the unperturbed solution on the separatrice obtained from (2.4) with $k = 1$.

Note that the function which solves the problem of the existence of a homoclinic structure can be found as the limit L of the function $2\pi pA(-\beta t'; h_{p1})$ as $h_{p1} \Rightarrow 1$, where $-\beta t'$ replaces the argument $p\upsilon$ in A. The latter is connected with the fact that the integrands in the definition of A (see formula (2.2)) and in the definition of the Mel'nikov function have the same form with several different arguments ($\beta(t - t')$ in Δ_1 and $t + p\upsilon/q\beta$ in A). These functions obviously contain the same harmonics.

The limit L can be calculated either as $h \Rightarrow 1 - 0$ (from the oscillatory region, we denote it by L^-), or as $h \Rightarrow 1 + 0$ (from the rotational region, we denote it by L^+). In our case $L^- = 2L^+$. A similar relation is connected with the fact that in the unperturbed equation the limit of the oscillatory region immediately gives two branches of the separatrice (a separatrice contour), whereas the limit of the rotational region only gives one separatrice loop. Then, in the autonomous equation (1.4) ($C_1 = 0$) we can determine the continuous global generating Poincaré–Pontryagin function for all values of $h \in (-1, \infty)$.

The quantity E represents the value of the splitting of the separatrice in the conservative case, and also in the non-conservative case when $C_2 = -C_3/3$. When $|E| = |E_*|$ we have (up to terms $O(\varepsilon)$) contact between the corresponding separatrices and the fixed point $(\pi, 0)$.

Note that here, as for Duffing's equation [11], a non-trivial hyperbolic set appears just before the instant of contact. When the condition $C_2 = -C_3/3$, $C_2^2 + C_3^2 \neq 0$ is satisfied in the perturbed autonomous system ($C_1 = 0$) the limit cycle forms a separatrice contour. In a non-autonomous system ($C_1 \neq 0$) in this case a quasi-attractor exists [16] if $\varepsilon\sigma^{(s)}(1) = \varepsilon(C_2 - C_3) < 0$.

Note that the "amplitude" E decreases exponentially as the frequency increases. Hence, the width (with respect to the energy h) of the neighbourhood containing the quasi-attractor decreases rapidly as β increases. When $|E| < |E_*| + O(\varepsilon)$ the separatrices of the fixed point $(\pi, 0)$ do not intersect. However, they intersect with the separatrices of the hyperbolic periodic (fixed) points situated in the neighbourhoods of the split resonance levels (if such exist).

It is this situation that is shown in Fig. 1(a). Similar intersections of the separatrices give heteroclinic points. In Fig. 1(b) we show a "quasi-attractor" ($B^{(1)}(1) = B^{(2)}(1) = 0$), obtained on a computer for Eq. (1.3) with $p_1 = 0.1$, $p_2 = 1/3$, $p_3 = -0.1$; $\beta = 1.6$. This quasi-attractor contains about 5000 iterations of one initial point. The fairly large size of the "width" if the neighbourhood of the quasi-attractor for a small amplitude p_1 of the external force is particularly noteworthy (compare with [16]). In Fig. 1(b) we also show a partially traversed resonance with $p = 2$ and $q = 1$. The regions of attraction of the resonance mode is very thin (it merges with the spiral lines in Fig. 1(b). Hence, if we take the initial point in the neighbourhood of the origin of coordinates, then, with a probability close to unity, an irregular mode corresponding to the quasi-attractor will be established in the system.

5. CONCLUSION

Formulae (2.6)–(2.7) enable us to obtain quantitative estimates of the existence of any resonance mode and the position of the corresponding resonance zone. The number of split resonances when $C_2^2 + C_3^2 \neq 0$ is limited. For actual motions of the pendulum (1.3), when small non-conservative forces are present, there will most probably be a single resonance mode with $p = 2$ and $q = 1$ in the oscillatory region, and in the rotational region with $p = 1$ and $q = 1$.

Note also the following features in the investigation of Eqs (1.3) and (1.4).

1. The change from Fig. 3(a) to Fig. 3(c) corresponds to two bifurcations of "period doubling", where the transition from parametric resonance (Fig. 3b) to ordinary non-linear resonance (Fig. 3c) is related to the bifurcation of the production from a complex fixed saddle point of two periodic saddle points (the period is equal to two) and a node (a focus).

2. The bifurcation of the quasi-attractor appearing in the neighbourhood of the unperturbed separatrice (Fig. 1b) is the most interesting. It occurs for any amplitude of the external force (the parameter C_1): it is sufficient solely that $B^{(s)}(1) = 0$, $(C_2 = -C_3\sqrt{3})$, $\varepsilon(C_2 - C_3) < 0$ for example, $C_2 = -1/30$, $C_3 = 0.1$, $\varepsilon > 0$.

3. No resonance with $q > 1$ and odd p in the oscillatory region or resonances with $q > 1$ and even p in the rotational region occur in the quasi-integrable non-conservative case.

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REFERENCES.

1. STRUBLE R. A., Oscillations of a pendulum under parametric excitation. *Q. Appl. Math.* **21**, 2, 121–131, 1963.
2. STRUBLE R. A. and MARLIN J. A., Periodic motion of a simple pendulum with periodic disturbance. *Q. J. Mech. Appl. Math.* **18**, 4, 405–417, 1965.
3. BOLOTIN V. V., *Dynamical Stability of Elastic Systems*. Gostekhizdat, Moscow, 1956.
4. ZASLAVSKII G. M., and CHIRIKOV B. V., Stochastic instability of non-linear oscillations. *Usp. Fiz. Nauk* **105**, 1, 3–39, 1971.
5. VOLOSOV V. M. and MORGUNOV B. I., *Methods of Averaging in the Theory of Non-linear Oscillatory Systems*. Izv. MGU, Moscow, 1971.
6. MOROZOV A. D., *Systems, Close to Non-linear Integrable*. Izd. Gor'k. Univ., Gor'kii, 1983.
7. MOROZOV A. D. and SHIL'NIKOV L. P., Non-conservative periodic systems close to two-dimensional Hamiltonian. *Prikl. Mat. Mekh.* **47**, 3, 385–394, 1983.
8. MOROZOV A. D. On the global behaviour of self-oscillatory systems. *Int. J. Bifurcation and Chaos*. **3**, 1, 195–200, 1993.
9. MOROZOV A. D., The qualitative behaviour of solutions in the neighbourhood of the non-linear centre of two-dimensional time-periodic systems close to Hamiltonian. In *Applied Problems in the Theory of Oscillations*. Izd. Gor'k. Univ., Gor'kii, 1990.
10. ARNOL'D V. I., *Additional Chapters on the Theory of Ordinary Differential Equations*. Nauka, Moscow, 1978.
11. MOROZOV A. D., A complete qualitative investigation of Duffing's equation. *Diff. Urav.* **12**, 241–255, 1976.
12. POINCARÉ A., *Selected Papers*. Vol. 2. *New Methods of Celestial Mechanics*. Nauka, Moscow, 1972.
13. BIRKHOFF G. D., *Collected Mathematical Papers*, Vol. 2. Amer. Math. Soc., New York, 1950.
14. CHERRY T. M., The asymptotical solutions of the analytical Hamiltonian systems. *J. Different. Equat.* **4**, 2, 142–156, 1969.
15. KOZLOV V. V., Integrability and non-integrability in Hamiltonian mechanics. *Usp. Mat. Nauk* **38**, 1, 3–67, 1983.
16. MOROZOV A. D., Limit cycles and chaos in pendulum-type equations. *Prikl. Mat. Mekh.* **53**, 5, 721–730, 1989.
17. MOROZOV A. D., Resonance and chaos in parametric systems. *Prikl. Mat. Mekh.* **58**, 3, 41–51, 1994.
18. MEL'NIKOV V. K., The stability for the centre for time-periodic perturbations. *Trudy Mosk. Mat. Obshchestva* **12**, 3–51, 1963.

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